FRACTAL DIMENSIONS
AND CORRECTIONS TO SCALING
FOR CRITICAL POTTS CLUSTERS

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Abstract

Renormalization group and Coulomb gas mappings are used to derive theoretical predictions for the corrections to the exactly known asymptotic fractal masses of the hull, external perimeter, singly connected bonds and total mass of the Fortuin-Kasteleyn clusters for two-dimensional q-state Potts models at criticality. For $q = 4$ these include exact logarithmic (as well as $\log \log$) corrections.

1. INTRODUCTION

q-state Potts models, with interaction $-J\delta_{\sigma_i \sigma_j}$ ($\sigma_{i,j} = 1, 2, ..., q$) for the nearest neighbor (nn) sites i, j, have played an important role in condensed matter physics[1]. Here we study geometrical aspects of the critical Potts clusters, in two dimensions. For an arbitrary configuration of Potts states, one creates bonds between neighboring sites which have the same

state, $\sigma_i = \sigma_j$, with a probability $p = 1 - \exp\left(-J/kT\right)$. No bonds are created between sites with $\sigma_i \neq \sigma_j$. Here we study the fractal geometry at T_c of the clusters, made of sites connected by bonds[2]. Specifically, we measure the fractal dimensions D_M, D_H, D_{EP}, and D_{SC} describing the scaling of the cluster's mass, hull, external accessible perimeter[3] and singly connected bonds[4], respectively, with its radius of gyration R. As emphasized by Coniglio[5], many of these fractal dimensions have been derived analytically[6]. Some others have been found more recently[7,8]. These theoretical values are summarized in Table 1, in terms of the Coulomb gas coupling constant $g = \frac{4}{\pi} \arccos(-\frac{\sqrt{q}}{2})$.

Table 1: Exact theoretical predictions.

	D_S	$q=1$	$q=2$	$q=3$	$q=4$	c_S/a
g		$\frac{8}{3}$	3	$\frac{10}{3}$	4	
M	$(g+2)(g+6)/(8g)$	$\frac{91}{48}$	$\frac{15}{8}$	$\frac{28}{15}$	$\frac{15}{8}$	$\frac{1}{16}$
H	$1+2/g$	$\frac{7}{4}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{3}{2}$	$-\frac{1}{4}$
EP	$1+g/8$	$\frac{4}{3}$	$\frac{11}{8}$	$\frac{17}{12}$	$\frac{3}{2}$	$\frac{1}{4}$
SC	$(3g+4)(4-g)/(8g)$	$\frac{3}{4}$	$\frac{13}{24}$	$\frac{7}{20}$	0	-1
θ	$4(4-g)/g$	2	$\frac{4}{3}$	$\frac{4}{5}$	0 (log)	
θ'	$4/g$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{6}{5}$	1	
θ''	$2/g$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{1}{2}$	

We are currently studying the geometry of such Potts critical clusters, using numerical Monte Carlo simulations[9]. These simulations show that the asymptotic power law dependence of the various masses on R is approached relatively slowly, and therefore the analysis of the data must include *correction terms*, particularly as q approaches 4. The present paper contains a brief summary of analytic results for several of these corrections. Our predictions for the correction exponents (apart from the analytic ones) are also listed in Table 1.

2. RENORMALIZATION GROUP

The first correction relates to the *dilution field* ψ, which is generated under renormalization even when one starts with the non-dilute case[10]. For our non-dilute case, one expects $\psi(\ell)$ to increase under the renormalization group recursion relations (RGRR's) from $-\infty$ towards its critical value ψ^* (≤ 0 for $q \leq 4$), while the cluster linear size, like all other lengths, rescales as $R(\ell) = Re^{-\ell}$. Following Cardy *et al.*[10], we assume that ℓ_0 prefacing iterations bring ψ from $-\infty$ up to $\psi(\ell_0) = \psi_0$, with $|\psi_0| \ll 1$. We then expand the RGRR for ψ in powers of ψ and $\epsilon = q - 4$,

$$\frac{d\psi}{d\ell} = a(\epsilon + \psi^2 + b\psi^3 + r\epsilon\psi + ...). \tag{1}$$

At $q = 4$, this yields

$$a(\ell - \ell_0) \approx \psi_0^{-1} - \psi(\ell)^{-1} + b\log\frac{b + \psi(\ell)^{-1}}{b + \psi_0^{-1}}. \tag{2}$$

Iterating up to $R(\ell^{\times}) = 1$, we can express $\psi(\ell^{\times})$ in terms of $\log R = \ell^{\times}$. For large R, $\psi_0/\psi(\ell^{\times}) \approx A(\log R + B\log(\log R) + E) + O(\log\log R/\log R)$, where $A = -a\psi_0$, $B = -b/a$ and E also depends on a, b, ℓ_0 and ψ_0.

For $q < 4$, an expansion to second order in $\epsilon' = \sqrt{-\epsilon}$ yields $\psi(\ell^{\times}) \approx \psi^* + \tilde{B}R^{-\theta}$, with $\psi^* = -\epsilon'(1 - (r - b)\epsilon'/2) + \ldots$, $\theta = 2a\epsilon'(1 - b\epsilon') + \ldots$ and $\tilde{B} \propto (\psi_0 - \psi^*)$. To order-$\epsilon'$, one obtains a full solution,

$$\psi(\ell) = -\epsilon'\frac{1 + \hat{B}e^{-\theta\ell}}{1 - \hat{B}e^{-\theta\ell}}, \tag{3}$$

where $\hat{B} = (\psi_0 + \epsilon')/(\psi_0 - \epsilon')$. Indeed, ψ approaches ψ^* for large ℓ.

To obtain the scaling of $M_S(R)$, we write the RGRR for the field h_S conjugate to the density $\rho_S \equiv M_S/R^d$ as

$$\frac{dh_S}{d\ell} = (y_S + c_S\psi(1 + e_S\psi + f_S\psi^2 + \ldots))h_S, \tag{4}$$

where the coefficients may depend on ϵ. ρ_S is then found as a derivative of the free energy with respect to h_S. For $q = 4$, its singular part becomes

$$\begin{aligned}
\rho_S(\ell) &\propto& e^{-d\ell}h_S(\ell)/h_0 \\
&=& \exp[(y_S - d)\ell + \int_{\ell_0}^{\ell}(c_S\psi(1 + e_S\psi + f_S\psi^2 + \ldots))d\ell] \\
&\propto& e^{(y_S-d)\ell}[\psi(\ell)/\psi_0]^{c_S/a}(1 + O(\psi(\ell))).
\end{aligned} \tag{5}$$

For large $\log R = \ell^{\times}$, this becomes

$$M_S \propto R^{D_S}(\log R + B\log(\log R) + E)^{-c_S/a}(1 + O(\log\log R/\log R)), \tag{6}$$

with $D_S = y_S(q = 4)$, and c_S/a is to be taken from Table 1 (see below) . Note that $B = -b/a$ is universal (i. e. independent of ψ_0), and the non-universal constant E is the same for all S. Equation (6) generalizes the logarithmic corrections of Cardy *et al.*[10].

In practice, the numerical results are always analyzed by looking at the local logarithmic slope,

$$\begin{aligned}
D_S^{\text{eff}} &=& d\log M_S/d\log R = d\log h_S/d\ell|_{\ell=\ell^{\times}} \\
&=& y_S + c_S\psi(\ell^{\times})(1 + e_S\psi(\ell^{\times}) + f_S\psi(\ell^{\times})^2 + \ldots).
\end{aligned} \tag{7}$$

In some cases, this expression (in which $\psi(\ell^{\times})$ is related to $\log R = \ell^{\times}$ via Eq. (2)) gave a better fit than the derivative of the approximate expression in Eq. (6).

For $q < 4$, to leading order in ϵ', the same procedure turns Eq. (3) into

$$M_S \propto R^{D_S}(1 - \hat{B}R^{-\theta})^{-c_S/a} \approx R^{D_S}(1 + f_S R^{-\theta}), \tag{8}$$

where $D_S \approx y_S - c_S\epsilon'$ and $\theta \approx 2a\epsilon'$. The RHS of this equation remains correct also for higher orders in ϵ'. Note that to the lowest order in ϵ', the ratios $f_S/f_{S'}$ are universal, being equal to $c_S/c_{S'}$. This is similar to analogous ratios for thermodynamic properties in the usual ϵ-expansion[11]. Expanding the exact D_S (Table 1) in ϵ' yields c_S. Using also $a = 1/\pi$

(see below) yields our predictions for c_S/a (given in Table 1), to be used in fitting Eq. (6). The form on the RHS of Eq. (8) is already implied by den Nijs[12], who found that the pair correlation functions $G_H(r)$ can be expanded as a sum over r^{-2x_n}, implying a leading correction exponent $\theta = 2(x_{n+1} - x_n) = 4(4 - g)/g$. Expanding this expression in powers of ϵ' yields the coefficients $a = 1/\pi$ and $b = -1/2\pi$, which we use in our fits to Eq. (2). The value $a = 1/\pi$ also agrees with Cardy *et al.*[10]. This expression for θ also reproduces known results for $q = 2, 3$, as listed in Table 1.

3.   COULOMB GAS

The second source of corrections involves new contributions to the relevant pair correlation functions in the Coulomb gas representations[12]. In some of the exact derivations, the q-state Potts model renormalizes onto the vacuum phase of the Coulomb gas, involving 'particles' with electric and magnetic 'charges' (e, m). At criticality, the corresponding Coulomb gas has a basic 'charge' $\phi = |2 - g/2|$ mod 4. Various Potts model two-point correlation functions $G_S^P(\vec{r})$ are then mapped onto Coulomb gas analogs, which give the probability of finding two charged particles at a distance r apart. Asymptotically, these are given by

$$G^{CG}_{[(e_1,m_1),(e_2,m_2)]}(\vec{r}) \propto r^{-2x^{CG}_{[(e_1,m_1),(e_2,m_2)]}}, \tag{9}$$

where

$$x^{CG}_{[(e_1,m_1),(e_2,m_2)]} = -\frac{e_1 e_2}{2g} - \frac{g m_1 m_2}{2}. \tag{10}$$

Hence one identifies $D_S = d - x^{CG}_{[(e_1,m_1),(e_2,m_2)]}$, with $d = 2$. The results in Table 1 for $S = M$ were obtained by den Nijs[12], who noted that the spin-spin correlation function of the Potts model maps onto a Coulomb gas total electric charge $Q = -2\phi$, which splits into the two charges $e_{1,2} = \pm 1 - \phi$ (and $m_{1,2} = 0$). Continuing along similar routes, Saleur and Duplantier[6] used a mapping onto the body-centered solid-on-solid model, requiring a vortex-antivortex pair with $e_{1,2} = -\phi$ and $m_{1,2} = \pm 1/2$ or ± 1 for the fractal dimensions of $S = H$ or $S = SC$. The Table also contains Duplantier's recent result[8] for $D_{EP} = 2 - x_{EP}^P$, which has not been expressed in terms of Coulomb charges. The results for x_H^P and x_{SC}^P are special cases of the expression $x_\ell = g\ell^2/32 - (4 - g)^2/(2g)$, with $\ell = 2$ and 4 respectively[6]. For percolation ($q = 1$ and $g = 8/3$), this expression also yields $x_{EP}^P = x_3 = 2/3$ for the external perimeter and $x_G^P = x_6 = 35/12$ for the gates to fjords.[7]

   We now turn to corrections to the leading behavior. den Nijs[12] derived such corrections for the order parameter correlation function. In that case, he noted that the charge $Q = -2\phi$ could also split into the pair $e_{1,2} = \pm 3 - \phi$, yielding a contribution to G_M^P of the form $r^{-2x_{M,2}^P}$, with $x_{M,2}^P = x^{CG}_{[(3-\phi,0),(-3-\phi,0)]} = x_M^P + 4/g$. Since $D = d - x$ usually represents a fractal dimension, we relate each of these correction terms to some subset of the cluster, with dimension $D_{M,2} = 2 - x_{M,2}^P = D_M - 4/g$. Writing M_M as a sum of powers R^{D_i},[13] we have $M_M \propto R^{D_M}(1 + f'R^{-\theta'})$, with $\theta' = 4/g$.

   As far as we know, there has been no discussion of the analogous corrections to the other subsets discussed here. In the spirit of den Nijs[12], we note that the correlation function for both H and SC could also result from electrical charges $e_{1,2} = \pm 2 - \phi$, instead of

$-\phi$. For both of these cases this would give $x' = x + 2/g$, hence a correction exponent $\theta'' = 2/g$. At the moment, there exists no theory for corrections to M_{EP}. However, in the spirit of the renormalization group it is also reasonable to interpret θ' and θ'' as the scaling exponent of some irrelevant perturbation (yet to be identified). If that were true then we might expect the same perturbation also to affect other quantities, like M_{EP}. This conjecture is supported by the 'superuniversal' relation, $(D_H - 1)(D_{EP} - 1) = 1/4$, found by Duplantier[8]. If this relation also holds for the effective dimensions (as happens e. g. in the ϵ-expansion[11]), then H and EP should have the same correction exponents.

4. ANALYTIC CORRECTIONS; SUMMARY

The last source of corrections involves 'analytic' terms, coming e. g. from linear cuts with dimensions $(D_S - 1)$, [13] or from replacing R by $(R + a)$, since there are many possible candidates for the correct linear measure of the cluster. These would imply corrections of relative size $1/R$.

Combining all of these sources, we end up with the prediction (for $q < 4$)

$$D_S^{\text{eff}} = D_S + \sum_i f_i R^{-\theta_i}, \tag{11}$$

with $\theta_i = \theta$, θ' (or θ'') and 1. Indeed, our numerical simulations[9] basically confirm these expressions.

In summary, we have presented several general expressions for the q-dependent corrections to the asymptotic R-dependence of the mass, hull, external perimeter and singly connected bonds. Such corrections are crucial for fitting numerical data. It would be nice to have a unifying theory, which would confirm these expressions in a rigorous way. It would also be nice to obtain similar corrections for other geometrical and physical quantities, e. g. the number of gates to fjords[7].

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